Robust adaptive neural network control of a class of uncertain strict-feedback nonlinear systems with unknown dead-zone and disturbances

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\textbf{ABSTRACT}

In this paper, a robust adaptive neural control design approach is presented for a class of perturbed strict-feedback nonlinear systems with unknown dead-zone. In the controller design, different from existing methods, all the virtual control laws need not be actually implemented at intermediate steps, and only one actual robust adaptive control law is constructed by approximating the lumped unknown function of the system with a single neural network at the last step. By this approach, the structure of the designed controller is much simpler since the causes for the problem of complexity growing in existing methods are eliminated. Stability analysis shows that the proposed scheme can guarantee the uniform ultimate boundedness of all the closed-loop system signals, and the steady-state tracking error can be made arbitrarily small by appropriately choosing control parameters. Simulation studies demonstrate the effectiveness and merits of the proposed approach.

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1. Introduction

Adaptive backstepping has been a powerful method for synthesizing controllers for lower-triangular nonlinear systems with uncertainties. In [1–4], several adaptive backstepping control design methods were developed for strict-feedback and pure-feedback nonlinear systems with linearly parameterized uncertainties. For nonlinear systems with uncertainties that cannot be linearly parameterized or are completely unknown, online approximation based adaptive control techniques have been found to be particularly useful, such as neural networks (NN) control [5–20], and fuzzy control [21–32]. In [5–10,28,30–32], online approximation based adaptive backstepping control methods were presented for uncertain strict-feedback nonlinear systems. While in [11–18,22], adaptive backstepping control schemes were developed for uncertain pure-feedback nonlinear systems by using online approximation techniques.

A drawback exists in aforementioned adaptive control design methods. That is, the complexity of the designed controller grows drastically as the system order increases due to two reasons. One is the repeated differentiations of certain nonlinear functions in the controller design process, and the other one is the use of multiple online approximators. It is difficult to implement a controller if the controller is complex. To deal with the complexity growing problem, a dynamic surface control (DSC) technique was proposed by introducing a first-order filter of the synthetic input at each step in the traditional backstepping approach [33]. In [34], by incorporating the DSC technique into the online approximation based adaptive control design framework, an adaptive backstepping DSC approach was presented for a class of uncertain strict-feedback nonlinear systems. In recent years, a lot of online approximation based adaptive DSC design methods were developed for uncertain nonlinear systems in strict-feedback and pure-feedback forms, such as [35–44], and some references therein.

By using the DSC technique, the repeated differentiations of certain nonlinear functions in traditional design process were avoided. That is, one of the reasons behind the complexity growing problem is eliminated. However, the complexity growing problem is not solved completely because many online approximators are employed in these designs. The use of too many approximators makes the complexity of the both controller structure and computation grow significantly. To eliminate the two reasons behind the complexity growing problem simultaneously, single neural network (SNN) approximation based adaptive control design methods were
developed in [45,46]. In these methods, the designed controllers can be given directly, and the structures of the controllers are simpler.

Dead-zone nonlinearity, which is a common nonsmooth nonlinear characteristic, exists widely in many components of the actuators such as valves, DC servo motors, and other devices. The presence of the characteristic may cause deterioration of the control system performance. For a long time, the study about the dead-zone nonlinearity has been drawing much interest of the researchers [28,30,32,47,52,35]. For some classes of systems with unknown dead-zones, adaptive dead-zone inverse methods were developed in [47–52]. In these methods, the designed controllers can be given directly, and the structures of the controllers are simpler. Appropriately choosing control parameters, simulation results demonstrate the effectiveness of the approach.

Section 3 proposes the SNN problem formulation and preliminaries. Some assumptions and conditions are given in this section. Section 3.1 introduces the assumptions, and Section 3.2 presents the control objective for the system 1 such that all the signals in the close-loop system remain uniformly ultimately bounded, and the system output y follows the reference input signal y_r(t).

Assumption 1. The signs of g_i(x_i), i = 1, …, n, are known, and there exist positive constants g_0 and g_1 such that (i) |g_i(x_i)| ≥ g_0, ∀x_i ∈ R, and (ii) |g_i(x_i)| ≤ g_1, ∀x_i ∈ Δx_i ⊂ R, where Δx_i is a compact region.

Assumption 2. There exist smooth positive functions φ_i(x_i), i = 1, …, n, such that |Δ_i(x_i, t) ≤ φ_i(x_i), ∀(x_i, t) ∈ R^2 × R_+.

Remark 1. Assumption 2 implies that the class of uncertainties Δ_i(x_i, t), i = 1, …, n, satisfy a triangularity condition in terms of x_i. This will be exploited for the ease of the controller design. Similar assumptions to Assumption 2 are required in [9,10,35]. It is noted that the exact expressions of φ_i(x_i), i = 1, …, n, are not needed in the controller design process in this paper.

2.2. Dead-zone characteristic

The dead-zone characteristic of the actuator can be represented as follows [35]:

\[ u = D(v) = \begin{cases} D_1(v), & v \leq d_1, \\ 0, & d_1 < v < d_2, \\ D_2(v), & v \geq d_2, \end{cases} \]

where d_1 ≤ 0 and d_2 ≥ 0 are the unknown bounded constants, D_1(v) and D_2(v) are the unknown smooth functions.

Assumption 3. There exist positive constants D_{min}, D_{10}, D_{11}, D_{10} and D_{11} such that

\[ \begin{cases} 0 < D_{min} \leq D_1(v) \leq D_{10}, & \forall v \in (-\infty, d_1], \\ 0 < D_{10} \leq D_1(v) \leq D_{11}, & \forall v \in [d_1, +\infty), \end{cases} \]

where D_1(v) = dD_1(v)/dv and D_2(v) = dD_2(v)/dv.

Let \( D_{min} = \min(D_{min}, D_{10}) \) and \( D_{max} = \max(D_{10}, D_{11}) \). Using the mean value theorem, the dead-zone characteristic (2) can be rewritten as follows:

\[ u = D(v) = D(v) + d(v), \]

where

\[ D(v) = \begin{cases} D_1(d_2), & v \leq d_1, \\ D_2', & d_1 < v < d_2, \\ D_2(d_2), & v \geq d_2, \end{cases} \]

\[ d(v) = \begin{cases} -D_1'(d_1)d_1, & v \leq d_1, \\ -D_2', & d_1 < v < d_2, \\ -D_2'(d_2), & v \geq d_2, \end{cases} \]

2. Problem formulation and preliminaries

2.1. Systems description

Consider a class of uncertain nonlinear dynamical systems as follows:

\[ \begin{align*} \dot{x}_i &= g_i(x_i)x_i + f_i(x_i) + \Delta_i(x_i, t), & 1 \leq i \leq n - 1, \\ \dot{x}_n &= g_n(x_n)u + f_n(x_n) + \Delta_n(x_n, t), \\ u &= D(v), \\ y &= x_1, \end{align*} \]

where \( x_i = [x_1, \ldots, x_i]^T \in R^i, i = 1, \ldots, n \), are the system state variables; \( u \in R \) and \( y \in R \) are the system input and output, respectively; \( v \) is the output from the controller; the dead-zone characteristic of the actuator is described as \( D(v); f_i(x_i) \) and \( g_i(x_i), i = 1, \ldots, n \), are the unknown smooth nonlinear functions; and \( \Delta_i(x_i, t) \), \( i = 1, \ldots, n \), are the system uncertainties which could come from measurement noise, modeling errors, external disturbances, modeling simplifications or changes due to time variations, etc [10,31].

The control objective is, for a given reference input signal \( y_r(t) \), where \( y_r(t), y_r(t), \ldots, y_r(n)(t) \) are bounded for \( t \geq 0 \), to design a robust adaptive controller for the system (1) such that all the signals in the close-loop system remain uniformly ultimately bounded, and the system output \( y \) follows the reference input signal \( y_r(t) \).

Assumption 1. The signs of \( g_i(x_i), i = 1, \ldots, n \), are known, and there exist positive constants \( g_0 \) and \( g_1 \), such that (i) \( |g_i(x_i)| \geq g_0 \), \( \forall x_i \in R \), and (ii) \( |g_i(x_i)| \leq g_1 \), \( \forall x_i \in \Delta x_i \subset R \), where \( \Delta x_i \) is a compact region.

Assumption 2. There exist smooth positive functions \( \phi_i(x_i), i = 1, \ldots, n \), such that \( |\Delta_i(x_i, t) \leq \phi_i(x_i), \forall (x_i, t) \in R^2 \times R_+ \).

Remark 1. Assumption 2 implies that the class of uncertainties \( \Delta_i(x_i, t) \), \( i = 1, \ldots, n \), satisfy a triangularity condition in terms of \( x_i \). This will be exploited for the ease of the controller design. Similar assumptions to Assumption 2 are required in [9,10,35]. It is noted that the exact expressions of \( \phi_i(x_i), i = 1, \ldots, n \), are not needed in the controller design process in this paper.
where \( \xi_i \in (v, d_i) \), if \( v \leq d_i \); \( \xi_i \in (d_i, v) \), if \( v \geq d_i \); constant \( D_y \) satisfies \( D_{\max} \leq D_y \leq D_{\max} \).

From (3), (5) and (6), it can be obtained that \( D_{\min} \leq D(v) \leq D_{\max} \) and \( |d(v)| \leq d^* = \max(-D_{\min}d_i, D_{\max}d_i) \).

2.3. Notation

(i) \( \| \cdot \| \) denotes the Euclidean norm of a vector; \( \lambda_{\max} (\cdot) \) denotes the largest eigenvalue of a square matrix.

(ii) \( Y_{l1}^{n1} = (y_{11}, y_{12}, \ldots, y_{1n1}) \), \( i = 1, \ldots, n \).

(iii) \( C_{ij} = c_{i1}c_{j1} + \cdots + c_{i1}c_{j1} \), \( j \leq i \), where \( c_{j1}, i = 1, \ldots, i \), are constants.

For example \( C_{13} = c_{11}c_{31} + c_{12}c_{32} + c_{13}c_{33} \).

The following notation is frequently used in the process of subsequent controller design:

\[
C_{ij} = c_{i1}c_{j1} - 1.1 - 1. 
\]

(7)

2.4. Radial basis function neural networks

In this paper, a radial basis function (RBF) NN is employed to approximate the lumped unknown function of the system at the last step. Before introducing the control design method, the approximation property of the RBF network should be first recalled. The RBF network takes the form \( \theta^T \xi(x) \), where \( \theta \in \mathbb{R}^N \) (\( N \) is the number of network nodes) is called the weight vector and \( \xi(x) \in \mathbb{R}^N \) is a vector valued function defined in \( \mathbb{R}^N \). Denote the components of \( \xi(x) \) by \( \rho_i(x), i = 1, \ldots, N \), which are called basis functions. In this work, \( \rho_i(x) \) is chosen as the commonly used Gaussian function, which has the following form:

\[
\rho_i(x) = e^{-x^T \sigma_i \xi^2 / \sigma_i^2}, \quad i = 1, \ldots, N, 
\]

(8)

where \( \xi_i \in \mathbb{R}^N \) is a constant vector called as the center of the basis function, \( \sigma_i > 0 \) is a real number called as the width of the basis function, and \( \mu > 0 \) is the amplification factor of the basis function. Let \( \Omega \subset \mathbb{R}^N \) be a compact set, then there is an RBF neural network \( \theta^T \xi(x) \) which can approximate an unknown continuous real-valued function \( f(x) \) on the compact set \( \Omega \). According to the approximation property of the RBF network, we have that for any \( \varepsilon > 0 \), by properly choosing \( \mu, \sigma, \xi_i, \quad i = 1, \ldots, N, \quad \text{for some sufficiently large integer } N, \) there exists an ideal weight vector \( \theta \in \mathbb{R}^N \) such that the approximation error bounded by \( e^* > 0 \), i.e.,

\[
f(x) = \theta^T \xi(x) + \varepsilon, \quad x \in \Omega.
\]

(9)

where \( |\varepsilon| \leq \varepsilon \), where \( \varepsilon \) represents the network reconstruction error.

Since \( \theta \) is unknown, we need to estimate \( \theta \) on-line. The notation \( \theta \) is used to denote the estimation of \( \theta \) and an adaptive law will be developed to update it.

For RBF network, the following lemma provides an upper bound on the Euclidean norm of vector \( \xi(x) \), which is essential in proving our main result.

Lemma 1 (Wang [13], Kudela et al. [55]). Consider the RBF network. Let \( p = (1/2)\min_{x, \mu} \| \xi_i - \xi_j \| \), and let \( q \) be the dimension of input \( x, \mu \) and \( \sigma \) are the amplification factor and width of basis function, respectively. Then one has

\[
\| \xi(x) \| \leq \sum_{i=0}^p \sum_{i=0}^p 3\mu q(i+2)^{-1} e^{-2x^2 / \sigma_i^2} = \varepsilon.
\]

(10)

3. SNN based robust adaptive control design

In this section, an SNN based robust adaptive controller will be established for the class of uncertain strict-feedback nonlinear systems (1). At step \( i = 1, \ldots, n-1 \), a virtual robust feedback control law is given in which an unknown function is contained. At step \( n \), a desired robust control law is first given, and then an actual robust adaptive control law is implemented by replacing the lumped unknown function of the system with an SNN approximator.

Step 1: Let \( z_1 = x_1 - y \), The derivative of \( z_1 \) is

\[
\dot{z}_1 = g_1(x_1) + f_1(x_1) + \Delta_1(x_1, t), \quad \dot{y}_1 = g_1(x_1) + f_1(x_1) + \Delta_1(x_1, t),
\]

(11)

where \( f_1(x_1, y_1) = f_1(x_1) - \dot{y}_1 \) is an unknown smooth function.

Choose a Lyapunov function candidate as \( V_1 = (1/2)z_1^2 \). Using Assumption 2 and Young's inequality, we can obtain that

\[
\dot{V}_1 = z_1(g_1(x_1) + f_1(x_1) + \Delta_1(x_1, t)) \leq z_1(g_1(x_1) + f_1(x_1) + \Delta_1(x_1, t)) + \frac{1}{2} \leq z_1g_1(x_1) + f_1(x_1) + \Delta_1(x_1, t),
\]

(12)

where \( f_1(x_1, y_1) = (1/g_1(x_1))f_1(x_1) + z_1 \theta_1^2(x_1) \) is an unknown smooth function.

The virtual robust control law \( \alpha_2 \) is chosen as follows:

\[
\alpha_2 = -c_2z_1 - A_2(x_1, y_1),
\]

(13)

where \( c_1 \) is a positive real constant which will be specified later. Letting \( z_2 = x_2 - \alpha_2 \), we have

\[
\dot{V}_2 = -z_1^2 + z_1^2 + \Delta_1(x_1, t) = -z_1^2 + \Delta_1(x_1, t).
\]

(14)

Substituting \( \alpha_2 \) into \( z_2 \), we can obtain that

\[
z_2 = x_2 + c_1z_1 + f_1(x_1, y_1)^2 = x_2 - y_1 + C_{11}(x_1 - y_1) + f_1(x_1, y_1)^2,
\]

(15)

where \( f_1(x_1, y_1)^2 = f_1(x_1, y_1)^2 + \dot{y}_1 \) is an unknown smooth function.

Step 2: The derivative of \( z_2 \) is

\[
\dot{z}_2 = g_2(x_2) + f_2(x_2) + \Delta_2(x_2, t) - \dot{y}_2 + C_{11}(g_1(x_1) + f_1(x_1) + \Delta_1(x_1, t)) + \Delta_1(x_1, t) + \Delta_1(x_1, t)
\]

\[
+ \frac{\partial f_2}{\partial x_2}(g_1(x_1) + f_1(x_1) + \Delta_1(x_1, t))
\]

\[
+ \frac{\partial f_2}{\partial y_2}(g_1(x_1) + f_1(x_1) + \Delta_1(x_1, t))
\]

\[
= g_2(x_2) + f_2(x_2, y_2) + \Delta_2(x_2, t) + C_{11}(x_1 - y_1) + \Delta_1(x_1, t)
\]

(16)

where

\[
f_2(x_2, y_2) = f_2(x_2, y_2) + \Delta_2(x_2, t) + \Delta_2(x_2, t) - \Delta_2(x_2, t)
\]

\[
= g_2(x_2) + f_2(x_2, y_2) + \Delta_2(x_2, t) + C_{11}(x_1 - y_1) + \Delta_1(x_1, t)
\]

(17)

Choose a Lyapunov function candidate as \( V_2 = (1/2)z_2^2 \). Using Assumption 2 and Young's inequality, we can obtain that

\[
\dot{V}_2 = z_2^2 + z_2^2 + \Delta_2(x_2, t)
\]

\[
+ C_{11}(x_1 - y_1) + \Delta_1(x_1, t)
\]

\[
\leq z_2^2 + z_2^2 + \Delta_2(x_2, t) + \Delta_2(x_2, t) + \Delta_2(x_2, t)
\]

\[
+ \frac{1}{2} + \frac{1}{4} + \frac{1}{4}
\]

\[
= z_2^2 + z_2^2 + \Delta_2(x_2, t) + \frac{3}{2}
\]

(18)

where \( f_2(x_2, y_2) = (1/g_2(x_2))f_2(x_2, y_2) + z_2^2 \theta_2^2(x_2) + z_2^2 \theta_2^2(x_2) + \Delta_2(x_2, t) + \Delta_2(x_2, t) + \Delta_2(x_2, t) \) is an unknown smooth function.

The virtual robust control law \( \alpha_3 \) is chosen as follows:
where $c_1$ is a positive real constant which will be specified later. Letting $z_3 = x_3 - \alpha_{3, t}$, we have
\begin{equation}
\dot{V}_2 \leq -c_2g_i(x_2^3)z_2^2 + g_i(x_2)z_2z_3 + \frac{3}{4}.
\end{equation}
(19)
Substituting $\alpha_{3, t}$ into $z_3$ and using the notation (7), we can obtain that
\begin{equation}
z_3 = x_3 + \Delta_{3, t} + \mathcal{F}_3(x_2, y_2^3)
= x_3 - \tilde{y}_3 + \sum_{i=1}^{i-1} C_3\Delta_{i, t} - y_2^{i-1} + \mathcal{F}_3^2(x_2, y_2^3),
\end{equation}
(20)
where $\mathcal{F}_3(x_2, y_2^3) = c_2\mathcal{F}_3^2(x_1, y_1^{i+1}) + \mathcal{F}_3^2(x_2, y_2^3) + \tilde{y}_1$ is an unknown smooth function.
Step (3) $i \leq n - 1$; Consider
\begin{equation}
z_i = x_i - \tilde{y}_i^{i+1} + \sum_{j=1}^{i-1} C_{i-1,j} \Delta_{i-1,j} - \tilde{y}_i^{i-1} + \mathcal{F}_{i-1}(x_i, y_i^{i-1}).
\end{equation}
(21)
where $\mathcal{F}_{i-1}(x_i, y_i^{i-1})$ is an unknown smooth function. The derivative of $z_i$ is
\begin{equation}
\dot{z}_i = g_i(x_i)\Delta_{i, t} + \mathcal{F}_i(x_i, t) - \tilde{y}_i^{i+1}
+ \sum_{j=1}^{i-1} C_{i-1,j} \Delta_{i-1,j} - \tilde{y}_i^{i-1} + \mathcal{F}_{i-1}(x_i, t),
\end{equation}
(22)
where $\mathcal{F}_i(x_i, y_i^{i+1}) = f_i(x_i) - \tilde{y}_i^{i+1} + \sum_{j=1}^{i-1} C_{i-1,j} \Delta_{i-1,j} - \tilde{y}_i^{i-1} + \mathcal{F}_{i-1}(x_i, y_i^{i-1}) + \mathcal{F}_{i-1}(x_i, t) - \tilde{y}_i^{i+1} - \sum_{j=1}^{i-1} \frac{\partial F_{i-1}}{\partial y_i^{j}} \tilde{y}_i^{j+1} + \sum_{j=1}^{i-1} \frac{\partial F_{i-1}}{\partial \dot{y}_i^{j+1}} \dot{y}_i^{j+1}$ is an unknown smooth function.
Choose a Lyapunov function candidate as $V_i = (1/2)\dot{z}_i^2$. Using Assumption 2 and Young's inequality, we can obtain that
\begin{equation}
\dot{V}_i = \frac{1}{2} \dot{z}_i^2 + \sum_{j=1}^{i-1} C_{i-1,j} \Delta_{i-1,j} - \tilde{y}_i^{i-1} + \mathcal{F}_{i-1}(x_i, t),
\end{equation}
(23)
where $\mathcal{F}_i(x_i, y_i^{i+1}) = (1/2)g_i(x_i)[F_i(x_i, y_i^{i+1}) + \dot{y}_i^{i+1}] + \sum_{j=1}^{i-1} C_{i-1,j} \Delta_{i-1,j} - \tilde{y}_i^{i-1} + \mathcal{F}_{i-1}(x_i, y_i^{i-1})$ is an unknown smooth function.
The virtual robust control law $\alpha_{i+1, t}$ is chosen as follows:
\begin{equation}
\alpha_{i+1, t} = -c_2\Delta_i - \mathcal{F}_i(x_i, y_i^{i+1}),
\end{equation}
(24)
where $c_1$ is a positive real constant which will be specified later. Letting $z_{i+1} = x_{i+1} - \alpha_{i+1, t}$, we have
\begin{equation}
\dot{V}_1 \leq -c_2g_i(x_2)\tilde{y}_2^2 + g_i(x_2)\tilde{y}_2z_{i+1} + \frac{2i-1}{4}.
\end{equation}
(25)
Substituting $\alpha_{i+1, t}$ into $z_{i+1}$ and using the notation (7), we can obtain that
\begin{equation}
z_{i+1} = x_{i+1} + c_2\Delta_i + \mathcal{F}_i(x_i, y_i^{i+1})
= \sum_{j=1}^{i} \frac{1}{C_{i,j}} \Delta_{i,j} - y_i^{i+1} + \mathcal{F}_{i-1}(x_i, y_i^{i-1}) + \mathcal{F}_i(x_i, y_i^{i+1}),
\end{equation}
(26)
where $\mathcal{F}_i(x_i, y_i^{i+1}) = c_2\mathcal{F}_i^2(x_1, y_1^{i+1}) + \mathcal{F}_i^2(x_i, y_i^{i+1}) + \tilde{y}_i$ is an unknown smooth function.
Step n: The derivative of $z_n$ is
\begin{equation}
\dot{z}_n = g_n(x_n)\Delta_n + \mathcal{F}_n(x_n, t) - \tilde{y}_n^{n+1}
+ \sum_{j=1}^{n-1} C_{n-1,j} \Delta_{n-1,j} - \tilde{y}_n^{n-1} + \mathcal{F}_{n-1}(x_n, t) + \sum_{j=1}^{n-1} \frac{\partial F_{n-1}}{\partial y_n^{j}} \tilde{y}_n^{j+1} + \sum_{j=1}^{n-1} \frac{\partial F_{n-1}}{\partial \dot{y}_n^{j+1}} \dot{y}_n^{j+1},
\end{equation}
(27)
where $\mathcal{F}_n(x_n, y_n^{n+1}) = f_n(x_n) - \tilde{y}_n^{n+1} + \sum_{j=1}^{n-1} C_{n-1,j} \Delta_{n-1,j} - \tilde{y}_n^{n-1} + \mathcal{F}_{n-1}(x_n, y_n^{n-1}) + \sum_{j=1}^{n-1} \frac{\partial F_{n-1}}{\partial y_n^{j}} \tilde{y}_n^{j+1} + \sum_{j=1}^{n-1} \frac{\partial F_{n-1}}{\partial \dot{y}_n^{j+1}} \dot{y}_n^{j+1}$ is an unknown smooth function.
Choose a Lyapunov function candidate as $V_n = (1/2)\dot{z}_n^2$. Using Assumption 2 and Young's inequality, we can obtain that
\begin{equation}
\dot{V}_n \leq \frac{1}{2} \dot{z}_n^2 + \sum_{j=1}^{n-1} C_{n-1,j} \Delta_{n-1,j} - \tilde{y}_n^{n-1} + \mathcal{F}_{n-1}(x_n, t),
\end{equation}
(28)
where $\mathcal{F}_n(x_n, y_n^{n+1}) = (1/g_n(x_n))(F_n(x_n, y_n^{n+1}) + \dot{y}_n^{n+1}) + \sum_{j=1}^{n-1} C_{n-1,j} \Delta_{n-1,j} - \tilde{y}_n^{n-1} + \mathcal{F}_{n-1}(x_n, t) + \sum_{j=1}^{n-1} \frac{\partial F_{n-1}}{\partial y_n^{j}} \tilde{y}_n^{j+1} + \sum_{j=1}^{n-1} \frac{\partial F_{n-1}}{\partial \dot{y}_n^{j+1}} \dot{y}_n^{j+1}$ is an unknown smooth function.
From Young's inequality, we have
\begin{equation}
z_n \leq \frac{1}{2} \dot{z}_n^2 + \frac{2n}{4}.
\end{equation}
(29)
Substituting inequality (29) into inequality (28), it can be obtained that
\[
V_n \leq z_n g_\eta(x_n) D(V) + z_n^2 g_\eta^2(x_n) D(V) F^2(x_n, \hat{y}^{(n)}) + \frac{g_\eta(x_n)}{4D(V)} + \frac{2n}{4}.
\]
(30)

The desired robust control law \(v^*\) is chosen as follows:
\[
v^* = - c_n z_n - z_n F^2(x_n, \hat{y}^{(n)}),
\]
(31)
where \(c_n\) is a positive real constant which will be specified later. Substituting \(z_n\) into \(v^*\) and using the notation (7), we can obtain that
\[
v^* = - c_n \left[ x_n - \hat{y}^{(n-1)} + \sum_{j=1}^{n-1} C_n x_{n-j} - \hat{y}^{(n-1-j)} + F^*_n (x_{n-1}, \hat{y}^{(n-1)}) \right]
- z_n F^2(x_n, \hat{y}^{(n)}),
\]
(32)
where
\[
F^*_n (x_n, \hat{y}^{(n)}) = c_n F^*_n (x_{n-1}, \hat{y}^{(n-1)}) + z_n F^2_n (x_n, \hat{y}^{(n)}),
\]
and \(F^2_n (x_n, \hat{y}^{(n)})\) is an unknown smooth function.

Given a compact set \(Q \subset R^{2n+1}\), and let \(\theta\) and \(\varepsilon\) be such that for any \((x_n, \hat{y}^{(n)}) \in Q\),
\[
F^*_n (x_n, \hat{y}^{(n)}) = \theta^T \xi(x_n, \hat{y}^{(n)}) + \varepsilon,
\]
with \(|\varepsilon| \leq \varepsilon^*\).

The actual robust adaptive control law is chosen as follows:
\[
v = - \sum_{j=1}^{n} C_n (x_{n-j} - \hat{y}^{(n-j)}) - \hat{\theta}^T \xi(x_n, \hat{y}^{(n)}),
\]
(33)
where \(\hat{\theta}\) is the estimation of \(\theta\) and is updated as follows:
\[
\hat{\theta} = \Gamma (\hat{\theta}^T - \eta \hat{\theta}),
\]
(35)
with a constant matrix \(\Gamma = \Gamma^T > 0\), and a real scalar \(\eta > 0\).

**Remark 2.** For the class of uncertain strict-feedback nonlinear system (1), the designed controller only contains an actual control law and an adaptive law, and can be given directly. Though the above design procedure contains \(n\) steps, the virtual control laws at intermediate steps are not necessary to be implemented. Thus, the structure of the designed controller is much simpler than that of exiting design approaches, and the controller realization is easier.

### 4. Stability analysis

In this section, we show that the actual control law and the adaptive law introduced in Section 3 can guarantee the UUB of all the closed-loop system signals.

From (31)–(34), we can obtain that
\[
v \leq - c_n \hat{\theta} - z_n F^2(x_n, \hat{y}^{(n)}) - \hat{\theta}^T \xi(x_n, \hat{y}^{(n)}) + \varepsilon,
\]
(36)
where \(\hat{\theta} = \theta - \hat{\theta}\). Substituting (36) into (30), we have
\[
V_n \leq g_\eta(x_n) D(V) [- c_n \hat{\theta} - z_n F^2(x_n, \hat{y}^{(n)}) + \xi(x_n, \hat{y}^{(n)}) + \varepsilon] + \frac{2n}{4D(V)} + \frac{2n}{4}.
\]
(37)

**Theorem 1.** Given \(\varepsilon^*\), let \(\theta \in R^n\) be such that (33) hold in the compact set \(Q \subset R^{2n+1}\) with \(|\varepsilon| \leq \varepsilon^*\). Consider the closed-loop system consisting of the system (1), the actual control law (34) and the adaptive law (35). Then, for any bounded initial conditions, all the signals in the closed-loop system remain uniformly ultimately bounded, and the steady state tracking error can be made arbitrarily small by appropriately choosing control parameters.

**Proof.** Consider the Lyapunov function candidate of the closed-loop system as \(V = \sum_{i=1}^{n} V_i + (1/2) \hat{\theta}^T \Gamma^{-1} \hat{\theta}\). The derivative of \(V\) is
\[
\dot{V} = \sum_{i=1}^{n} \left[ - c_i g_i(x_i) \xi_i^2 + g_i(x_i) \xi_i z_{i+1} + \frac{2n}{4D(V)} \right]
+ \frac{2n}{4D(V)} + \frac{2n}{4} + \hat{\theta}^T [z_i \xi(x_i, \hat{y}^{(i)}) - \eta \hat{\theta}].
\]
(38)
Using the facts that
\[
g_i(x_i) \xi_i z_{i+1} \leq \frac{g_i(x_i)}{2} z_i^2 + \frac{g_i(x_i)}{2} z_{i+1}^2, \quad i = 1, \ldots, n - 1,\]
\[
g_i(x_i) D(x_i) z_i \xi(x_i, \hat{y}^{(i)}) \leq \frac{2D_{\max}^2 g_i^2}{\eta} z_i^2 + \frac{\eta}{8} \hat{\theta}^T \hat{\theta},\]
\[
g_i(x_i) D(x_i) z_i \xi(x_i, \hat{y}^{(i)}) \leq \frac{D_{\max}^2 g_i^2}{\eta} z_i^2 + \frac{\eta}{8} \hat{\theta}^T \hat{\theta},\]
(39)
\[
z_i \hat{\theta}^T \xi(x_i, \hat{y}^{(i)}) \leq \frac{2\chi^2}{\eta} z_i^2 + \frac{\eta}{8} \hat{\theta}^T \hat{\theta},\]
(40)
we have
\[
\dot{V} \leq - \left( \frac{c_i g_i}{2} \frac{2\chi^2}{\eta} z_i^2 - \frac{n-1}{2} \sum C_n - \frac{g_i}{2} \frac{2\chi^2}{\eta} z_i^2 \right)
- \left( c_i g_i D_{\min} \frac{D_{\max}^2 g_i^2}{\eta} z_i^2 + \frac{D_{\max}^2 g_i^2}{\eta} z_i^2 + \frac{\eta}{4} \hat{\theta}^T \hat{\theta} + \frac{\eta^2 \hat{\theta}^T \hat{\theta}}{4} + \frac{n^2+1}{4} \right) + \frac{2n}{4D(V)}.
\]
(41)
Choosing
\[
\frac{c_i g_i}{2} \frac{2\chi^2}{\eta} z_i^2 \geq \frac{\alpha}{\eta} z_i^2, \quad i = 2, \ldots, n - 1,\]
\[
\frac{c_i g_i}{2} \frac{2\chi^2}{\eta} z_i^2 \geq \frac{\alpha}{\eta} z_i^2, \quad i = 2, \ldots, n - 1,\]
\[
\frac{c_i g_i D_{\min} \frac{D_{\max}^2 g_i^2}{\eta} z_i^2 + \frac{D_{\max}^2 g_i^2}{\eta} z_i^2 + \frac{\eta}{4} \hat{\theta}^T \hat{\theta} + \frac{\eta^2 \hat{\theta}^T \hat{\theta}}{4} + \frac{n^2+1}{4}}{\eta} \geq \frac{\alpha}{2},
\]
(41)
where \(\alpha > 0\), and letting
\[
\eta \leq \frac{\gamma}{\alpha z_i^2},
\]
(42)
we have
\[
\dot{V} \leq - \alpha V + \gamma V + \gamma V
\]
(43)
where \(\gamma = e^{\alpha z_i^2/4} + \eta \theta^2 / (n^2 + 1) / 4 + \frac{2}{4D(V)}.
\)
Solving the inequality (43) gives
\[
0 \leq V(t) \leq \frac{\gamma}{\alpha} (V(0) - \frac{\gamma}{\alpha} e^{-\alpha t}), \quad \forall t \geq 0.
\]
(44)

The inequality (44) means that \(V(t)\) eventually is bounded by \(\gamma/\alpha\). That is, \(z_i, i = 1, \ldots, n, \) and \(\hat{\theta}\) are uniformly ultimately bounded. Since \(x_i = z_i + y_i\) is uniformly ultimately bounded, the smooth function \(F_n(x_i, \hat{y}^{(i)})\) is also uniformly ultimately bounded. Then \(x_i = z_i + \alpha_t z_i = z_i - c_i z_i - F_n(x_i, \hat{y}^{(i)})\) is uniformly ultimately bounded accordingly. By analogy, \(x_3, \ldots, x_n\) are all uniformly ultimately
bounded. That is, all the closed-loop system signals are uniformly ultimately bounded. Moreover, by increasing the values of $c_i$, $i = 1, \ldots, n$, and reducing the value of $\lambda_{\max}(I^{-1})$, i.e., increasing the value of $\alpha$, the quantity $\gamma/\alpha$ can be made arbitrarily small. Thus, the tracking error $z_1$ can be made arbitrarily small. This concludes the proof. □

**Remark 3.** The control constants $c_1, c_2, \ldots, c_n$ determined by (41) are fairly conservative. They may be bigger than they really need. Eq. (41) just provides a guideline for choosing the control constants. In practice, these control constants can be adjusted according to the chosen neural network and other arguments.

5. Simulation examples

In this section, we will present two practical examples to demonstrate the effectiveness and merits of the proposed scheme.

**Reference signal processing:** The big differences $x_i - y_i^{(i-1)}$, $i = 1, \ldots, n$, may result in a very big control effort at the initial time. This is harmful to equipments in real systems. To avoid output saturation of the actuator at initial phase, some filters are used to make the reference signal smooth. The filters are given as follows:

$$T_i(t)\gamma_i^{(i-1)} + \gamma_i^{(i-1)} = y_i^{(i-1)}, \quad i = 1, \ldots, n,$$

(45)

where $\gamma_i^{(i-1)}$ and $y_i^{(i-1)}$ are the inputs and outputs of the filters, respectively; the initial conditions of the filters are $\gamma_i^{(i)}(0) = x_i(0)$; the filtering parameters are chosen as $T_i(t) = y_i e^{-\alpha t} + r_i$, where $y_i > 0$, $\alpha > 0$, $r_i > 0$.

**Remark 4.** The purpose of the aforementioned signal processing is to obtain a virtual reference signal which is tracked by the system output actually. Because the initial condition of the virtual reference signal is equal to the initial condition of the system states, all the error terms in the actual control law are small at the beginning of control. Furthermore, if the parameters $r_i$, $i = 1, \ldots, n$, are sufficiently small, the ultimate difference between the actual reference signal and the virtual reference signal is also sufficiently small. Thus, as long as the system output is capable of tracking the virtual reference signal very well, good actual tracking performance can be obtained accordingly.

5.1. Brusselator model

We firstly consider a very popular nonlinear oscillatory model of chemical kinetics as follows [56,17]:

$$\begin{align*}
\dot{x}_1 &= x_1^2 x_2 + A - (B + 1) x_1 + \Delta_1(x_2, t), \\
\dot{x}_2 &= (2 + \cos x_1) u + B x_1 - x_1^2 x_2 + \Delta_2(x_2, t), \\
y &= x_1,
\end{align*}$$

(46)

where $x_1$ and $x_2$ denote the concentrations of the reaction intermediates; positive constants $A$ and $B$ are parameters which describe the supply of "reservoir" chemicals; $\Delta_1(x_2, t)$ and $\Delta_2(x_2, t)$ are system uncertainties, which could come from external disturbances. This model was named as the Brusselator model which is introduced in [57] and studied in detail in [58].

According to the proposed approach in Section 3, the SNN based robust adaptive controller can be chosen as follows:

$$\begin{align*}
\dot{u} &= -C_{2,1}(x_2 - y_1) - C_{2,2}(x_1 - y_2) - \theta^T \xi(x_2, y_2), \quad (47) \\
\dot{\theta} &= \Gamma(x_1 \xi(x_2, y_2)) - \eta \dot{\theta}.
\end{align*}$$

In the simulation, it is assumed that $A = 1$, $B = 3$, $\Delta_1(x_2, t) = 0.7x_2^2 \cos(1.5t)$ and $\Delta_2(x_2, t) = 0$ [17]. The system initial condition is $[x_1(0), x_2(0)]^T = (2.7, 1)^T$. The reference signal is $y_r = 3 + \sin t + 0.5 \sin(1.5t)$.

The controller parameters chosen for simulation are $c_1 = 15$, $c_2 = 15$, $\Gamma = \text{diag}(4)$, $\eta = 0.2$, $N = 243$, $\sigma = 10$, $\mu = 2$. The centers of the Gaussian functions are $(1, 3.5) \times [-2, 0.2] \times (1, 3.5) \times [-2, 0.2] \times (-3, 0.3)$. The NN initial weight is $\theta(0) = 0$. The parameters used for reference signal processing are $\psi_1 = \psi_2 = 1$, $\omega_1 = \omega_2 = 1$, and $r_1 = r_2 = 0.01$.

Figs. 1-4 are the obtained simulation results in this case. From Fig. 1, it can be seen that fairly good tracking performance is obtained. The system output tracks the reference signal at a high precision. Fig. 2 gives the control signal of the closed-loop system. The update history of the neural network weight is given in Fig. 3. The performance of the system output tracking the virtual reference signal is shown in Fig. 4.

**Remark 5.** The above-mentioned Brusselator model is often used to test the performance of controller. Compared with controllers given in [56,17], the controller (47) contains only one actual input. Further investigations need to be made.

![Fig. 1. The performance of the system output tracking the reference signal.](image1)

![Fig. 2. Control input signal.](image2)

![Fig. 3. Norm of the network weight.](image3)
control law, therefore only one online approximator is actually needed in the controller implementation.

5.2. One-link manipulator with a BDC motor

Now, we give a popular benchmark of application example, i.e., the trajectory tracking control of a one-link manipulator actuated by a brush DC (BDC) motor, which has been considered in [39, 28]. The dynamics of a one-link manipulator actuated by a BDC motor can be expressed as follows:

\[
\begin{aligned}
E \dot{q} + B q + N \sin q &= I + \Delta L, \\
M \ddot{q} + H \dot{q} &= U - K_m \dot{q},
\end{aligned}
\]  

(48)

where \( q, \dot{q}, \) and \( \ddot{q} \) denote the link angular position, velocity and acceleration, respectively; \( I \) is the motor current; \( \Delta L \) represents the torque disturbance; and \( U \) is the input control voltage.

The control objective is to let the link angular position track a desired reference signal: \( y_r(t) = (\pi/2) (1 - e^{-0.1t}) \sin t \).

Letting \( x_1 = q, x_2 = \dot{q}, x_3 = I \) and \( u = U \), then Eqs. (48) can be expressed as

\[
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= g_2 x_3 + f_3(x_3) + \Delta_2(t), \\
\dot{x}_3 &= g_3 u + f_3(x_3),
\end{aligned}
\]  

(49)

where \( g_2 = 1/E, \ f_2(x_2) = -(N/E) \sin x_1 -(B/E)x_2, \ g_3 = 1/M, \ f_3(x_3) = -(K_m/M)x_2 -(H/M)x_3 \) and \( \Delta_2(t) = \Delta L/E \).

Same with [28], the dead-zone characteristic of the actuator is assumed as follows:

\[
u = D(v) = \begin{cases} 
1.3(v + 0.5), & v \leq -0.5, \\
0, & -0.5 < v < 0.4, \\
1.5(v - 0.4), & v \geq 0.4.
\end{cases}
\]  

(50)

The robust adaptive controller can be chosen directly according to the approach proposed in Section 3 as follows:

\[
\begin{aligned}
\nu &= -C_1 x_1 \bar{y}_x - C_2 x_2 \bar{y}_x - C_3 x_3 \bar{y}_x - \dot{\theta}^T \zeta(x_3, \bar{y}_r^T), \\
\dot{\theta} &= \Gamma (x_3 \bar{y}_x, \bar{y}_r^T) - \eta \dot{\theta}.
\end{aligned}
\]  

(51)

In the simulation, the parameter values of the appropriate units are the same as in [39] by \( E = 1, B = 1, M = 0.05, H = 0.5, N = 10 \) and \( K_m = 10 \). The torque disturbance is \( \Delta L = 4 \sin t \). The system initial condition is \( [x_1(0), x_2(0), x_3(0)]^T = [\pi/4, \pi/2, 0]^T \).

The parameters of the controller are chosen as \( c_1 = 15, c_2 = 15, c_3 = 15, \Gamma = \text{diag}(4), \eta = 0.2, N = 128, \sigma = 10, \) and \( \mu = 5 \). The centers of the Gaussian functions are \( [-2,2] \times [-2,2] \times [-2,2] \times [-2,2] \times [-2,2] \times [-2,2] \times [-2,2] \). The initial weight of the RBF network is \( \theta(0) = 0 \). The parameters used for reference signal processing are \( \psi_1 = \psi_2 = \psi_3 = 1, \omega_1 = \omega_2 = \omega_3 = 1, \) and \( \tau_1 = \tau_2 = \tau_3 = 0.01 \).

Fig. 4. The performance of the system output tracking the virtual reference signal.

Fig. 5. The performance of the link angular position tracking the desired reference signal.

Fig. 6. Control voltage and motor current.

Fig. 7. Norm of the network weight.
Figs. 5–8 are the obtained simulation results in this case. From Fig. 5, it can be seen that fairly good tracking performance is obtained. The link angular position tracks the desired reference signal at a high precision that the maximum steady state tracking error is about 3%. Fig. 6 gives the control voltage and the motor current of the actuator motor. The update history of the neural network weight is given in Fig. 7. The performance of the link angular position tracking the virtual reference signal is shown in Fig. 8.

6. Conclusion

A robust adaptive neural control design is developed for a class of uncertain strict-feedback nonlinear systems with dead-zone and disturbances. Though some virtual robust control laws are given at the intermediate steps of the design, only one actual robust adaptive control law needs to be implemented by replacing the lumped unknown function of the system with an SNN approximator at the last step. Thus, the designed controller only contains an actual control law and an adaptive law, and can be given directly. Compared with existing methods, the structure of the designed controller is much simpler, and the implementation of the controller is much easier in practice. Stability analysis shows that all the closed-loop system signals are uniformly ultimately bounded, and the steady state tracking error can be made arbitrarily small by adjusting the control parameters properly. The effectiveness and merits of the proposed approach are demonstrated by simulation results.

The problem of adaptive control is investigated for a class of single-input single-output (SISO) uncertain nonlinear systems in this paper. The study for a more general class of multi-input multi-output (MIMO) nonlinear systems is now under the authors’ considerations. In addition, the research of state observer based output feedback control has obtained much attention in recent years, and a lot of significative results, such as [24,25,29–31,44], are reported. The study of incorporating single approximator technique into a state observer based output feedback control design framework should be an interesting work.

References


