Formation control for nonlinear multi-agent systems by robust output regulation

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ABSTRACT

In this paper, the formation control problem of multi-agent systems with nonlinear dynamic is considered. Output regulation problem of nonlinear systems is generalized to the formation control problem of nonlinear multi-agent systems. The reference inputs or disturbance signals are generated by an exo-system, which can be considered as an active leader in the considered multi-agent systems. When some agents cannot obtain the information from the exo-system, the distributed nonlinear feedback controller is designed. Based on the internal model principle, the distributed output regulation formation control problem can be solved by solving the regulator equations. The illustrative example demonstrates the effectiveness of the main results.

1. Introduction

A multi-agent system (MAS) is a very complex system, made up of multiple interacting intelligent agents. Generally speaking, it is difficult for a single agent to finish some large or complex tasks; sometimes these tasks cannot be completed at all. However, multi-agent systems can cooperate to solve the problems that are beyond the capacities of any individual agent. More importantly, multi-agent systems have many advantages such as reliability, flexibility, reducing cost, improving system efficiency, providing some new capabilities and so on.

In recent years, cooperative control of multi-agent systems ([11–7] and the references therein) has been widely studied due to the development of advanced theory of complex systems and its broad applications in many fields. As one of the most important and fundamental problem of coordinated control, formation control of multi-agent systems has attracted much attention and has been widely applied in many fields recently, such as unmanned aerial vehicles (UAVS), autonomous underwater vehicles (AUVS), mobile robot systems (MRS) and so on. Formation control problem is to find a coordinated control scheme for multi-agent systems such that the agents would reach and maintain some desired formation or group configuration.

During the past decade, many different types of formation control methods have been proposed [8–12]: such as leader-following strategy, behavior-based approach and virtual structure method. In addition, some researchers also considered the formation control of multi-agent systems by other methods. Xiao et al. [13] applied the proposed nonlinear consensus protocol to the formation control, including time-invariant formation, time-varying formation and trajectory tracking. It turns out that all agents could achieve the expected formation after a finite time. Chen et al. [14] studied formation control design and stability analysis of double-integrator agents with directed communication links, in which the motions of the agents are restricted to given curves. In ref. [15], variants of a consensus algorithm were used to tackle the formation control problem of second-order multi-agent systems, and the formation control was proved to be asymptotically achieved. Chen et al. [16] studied the formation control problem for systems consisting of multi-agents that are described by first-order and second-order differential equations.

Moreover, since the 1990s, the output regulation problem for nonlinear systems has been studied (see, for example [17–20], and the references therein), owing to its profound theoretical background and practical values. The output regulation problem of a controlled system is that of controlling a plant to track (or reject) reference (or disturbance) signals, which are produced by the exo-system. When considering the tracking and formation problem of the multi-agent system under the framework of leader-following, the follower cannot completely get the information from the dynamic leader. It is difficult to consider the leader-following problem by the existing methods. However, from the perspective of output regulation, the problem will be solved easily. Recently, research on the output regulation problem for multi-agent systems

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has been investigated [21–24]. In [21], the authors formulated the formation control problem as a decentralized output regulation problem with the assumptions that all the agents can get the exogenous signals (leader). However, in practice, some agents cannot obtain the information of the leader, it is required to design a distributed controller, obtain their neighbor external measured state through the information exchange, and then design the distributed controller of the multi-agent system based on the measured state. A very recent work by F Hong and Huang et al. [22–24] designed the distributed controller to solve the output regulation problem of linear multi-agent system, and extended the results to the robust regulation problem of the multi-agent system with uncertainties. However, their studies are mainly about linear systems, the dynamics of the multi-agent being nonlinear is rarely seen, especially formation control by output regulation.

Motivated by the above discussions, we considered the formation control problem of a multi-agent system with nonlinear dynamic using robust output regulation. In other words, designing the controller such that the agents have to acquire a pre-defined geometric shape and track a reference trajectory while maintaining the formation. The reference or disturbance signals are generated by an exo-system, which can be considered as an active leader in the considered multi-agent systems. Based on the internal model principle, the formation control problem can be solved by solving the regulator equations.

The rest of this article is organized as follows. In Section 2, some model formulations and useful preliminaries are given. The main results of distributed robust formation output regulation problem for nonlinear multi-agent systems are discussed in Section 3. The numerical example is given to verify the theoretical results in Section 4. In the end, concluding remarks are provided in Section 5.

2. Problem formulation and preliminaries

Here, some preliminary knowledge of the algebraic graph theory is introduced for the following analysis (referring [25]). Let $G(\nu;\epsilon,A)$ be a weighted digraph of order $n$ with the set of nodes $\nu=\{1,2,\cdots,n\}$, set of arcs $\epsilon \subseteq \nu \times \nu$, and a weighted adjacency matrix $A=[a_{ij}] \in \mathbb{R}^{n \times n}$ with nonnegative elements. The node indexes belong to a finite index set $I=\{1,2,\cdots,n\}$. An arc of $I$ is denoted by $(ij)$, which starts from $i$ to $j$. The element $a_{ij}$ associated with the arc of digraph is positive, i.e. $a_{ij}>0$ if $(ij) \in \epsilon$. Moreover, we assume $a_{ii}=0$ for all $i \in I$. The set of neighbors of node $i$ is denoted by $N_i=\{(i,j) \in \epsilon: (j,i) \notin \epsilon\}$.

A diagonal matrix $D=\text{diag}(d_1,d_2,\cdots,d_n) \in \mathbb{R}^{n \times n}$ is a degree matrix of $G$, whose elements $d_i=\sum_{j \in N_i}a_{ij}$ for $i=1,2,\cdots,n$. Then the Laplacian matrix of the weighted digraph $G$ is defined as $L=D-A \in \mathbb{R}^{n \times n}$.

To study a leader-following problem, we also use another graph $\mathcal{G}$ which consists of $n$ agents and one leader (labeled by 0). For $\mathcal{G}$, if there is a path in $\mathcal{G}$ from each node $i$ in $\mathcal{G}$ to node 0, we say that node 0 is globally reachable in $\mathcal{G}$. Denote $b_i$ as the linked weight between agent $i$ and the leader and there are positive constants $\beta_i(i=1,2,\cdots,n)$ such that

$$b_i=\begin{cases} \beta_i, & \text{if the agent } i \text{ is connected to the leader;} \\ 0, & \text{otherwise.} \end{cases}$$

In this paper, the dynamics of the considered multi-agent system can be described as follows:

$$\begin{align*}
\dot{x}_i & = f_i(x_i, u_i, w, \mu), \\
w & = s(w), \\
y_0 & = q(w), \\
\dot{e}_i & = x_i - r_i - y_o, & i = 1,2,\cdots,N, \\
\end{align*}$$

where $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ represent the state and control input of the $i$-th agent, respectively, $w \in \mathbb{R}^n$ represents the unknown plant parameter and $w \in \mathbb{R}^n$ is an exogenous signal which may represent the reference signal to be tracked and (or) the disturbance to be rejected, and is generated by the exo-system $w = s(w)$. $y_0 \in \mathbb{R}^n$ is the measured output, and $r_i \in \mathbb{R}^n$ is the desired formation vector from the $i$-th agent to the leader. Here $e_i \in \mathbb{R}^n$ is the regulated output for the $i$-th agent, which describes the control target. It is assumed that $f_i(\cdot,\cdot), s(w), q(w)$ are sufficiently smooth and known $C^k(\geq 2)$ functions, and $s(0) = 0$, $q(0) = 0$.

In this paper, the exo-system can be considered as an active leader. In practice, only some of the agents can obtain signal of the exo-system. It also means that the state of the exo-system $w$ cannot be measured by all the agents, so it cannot be used in the control design. Moreover, $e_1$ may not be used directly too. Therefore, we introduce a virtual regulated error as follows:

$$e_\nu = \sum_{j \in N_i} a_{ij} (e_i - e_j) + b_i e_i$$

Now we define the dynamic feedback distributed controller based on the virtual regulated error as follows:

$$\begin{align*}
\dot{u}_i & = \hat{\theta}_i(z_i, e_v), \\
\hat{z}_i & = \eta_i(z_i, e_v),
\end{align*}$$

where $\hat{\theta}_i(\cdot,\cdot)$ and $\eta_i(\cdot,\cdot)$ are $C^k(\geq 2)$ functions. For convenience, we assume that $\hat{\theta}_i(0,0) = 0$ and $\eta_i(0,0) = 0$, $i = 1,2,\cdots,N$.

Remark 1. Different from conventional output regulation problem, distributed output regulation problem is mainly based on the virtual regulated error of the multi-agent system, and this method does not require each agent and the active leader to have identical dynamics. In [21], the author assumed that each agent can obtain the signal from the leader. In fact, each agent has to collect the information in a distributed way from its neighbor agents. In addition, the considered multi-agent systems in [22–24] are linear, while in this paper, the dynamic of the considered system is nonlinear, which can be more applicable in practice.

Denote

$$\begin{align*}
x &= (x_1, x_2, \cdots, x_N)^T, \\
\dot{x} &= (\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_N)^T, \\
\epsilon &= (e_1, e_2, \cdots, e_N)^T, \\
\epsilon_v &= (e_{v1}, e_{v2}, \cdots, e_{vN})^T, \\
f &= (f_1(\cdot), f_2(\cdot), \cdots, f_N(\cdot))^T, \\
\theta(z, \epsilon_v) &= (\theta_1(z_1, \epsilon_v), \theta_2(z_2, \epsilon_v), \cdots, \theta_N(z_N, \epsilon_v))^T, \\
\eta(z, \epsilon_v) &= (\eta_1(z_1, \epsilon_v), \eta_2(z_2, \epsilon_v), \cdots, \eta_N(z_N, \epsilon_v))^T.
\end{align*}$$

Then system (1) and controller (3) can be rewritten as

$$\begin{align*}
\ddot{\tilde{x}} &= f(\tilde{x} + \epsilon, u, w, \mu), \\
w &= s(w), \\
\epsilon &= \tilde{x} - I \otimes q(w),
\end{align*}$$

and

$$\begin{align*}
u &= \theta(z, \epsilon_v), \\
\dot{\tilde{z}} &= \eta(z, \epsilon_v).
\end{align*}$$

Denoting system (4) in the following form:

$$\begin{align*}
\ddot{x} &= f(x, u, w, \mu), \\
w &= s(w), \\
\epsilon &= x - I \otimes q(w),
\end{align*}$$

where we use the notation $\ddot{x}(\cdot)$ since we use $x$ instead of $x + \epsilon$ in $f(\cdot)$. Let $B_0 = \text{diag}(b_1, b_2, \cdots, b_N)$, then the virtual regulated error (2) can be rewritten as $e_v = Hx - (B_0 I_N) \otimes y_o$, where $H = (L + B_0) \otimes I_n$, $I_N = \{1,1,\cdots,1\}^T \in \mathbb{R}^n$. 

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Substituting (5) into (6) yields the closed-loop system
\[
\dot{x} = f(x, \theta(z, e_v), w, \mu),
\]
\[
z = \eta(z, e_v),
\]
\[
w = s(w),
\]
\[
\xi = x - 1 \odot q(w).
\]
(7)

As it is assumed that \( f(\cdot) \) is a sufficiently smooth and known \( C^k(k \geq 2) \) function, when considering the formation problem, the desired formation vector \( r_i \) is known, then the function \( f_i(\cdot) \) is also a sufficiently smooth and known \( C^k(k \geq 2) \) function vector. We assume that \( f_i(0, 0, 0, \mu) = 0 \).

Next, before presenting the main results, we will introduce some lemmas and definitions which play an important role in the proof of the main results.

**Lemma 1.** [17] Consider the system as follows:
\[
y = A_1 y + g_1(y, z),
\]
\[
z = A_2 z + g_2(y, z),
\]
where \( y \in \mathbb{R}^n, z \in \mathbb{R}^m \), \( A_1, A_2 \) are constant matrices and all the eigenvalues of \( A_1 \) have negative real parts, while all the eigenvalues of \( A_2 \) have zero real parts, \( g_1(\cdot) \) and \( g_2(\cdot) \) are \( C^2 \) mapping satisfying
\[
g_i(0, 0) = 0, \quad \frac{\partial g_i}{\partial y}(0, 0) = 0, \quad \frac{\partial g_i}{\partial z}(0, 0) = 0, \quad (i = 1, 2).
\]

Then there exist a constant \( \epsilon > 0 \) and a continuously differentiable function \( h(z) \) for all \( ||z|| < \epsilon \), such that \( y = h(z) \) is the center manifold system (8).

**Lemma 2.** [17] Suppose \( y = \pi(z) \) is a center manifold for the system (8) at (0, 0). Let \( (y(t), z(t)) \) be a solution curve of (8) with \( (y(0), z(0)) \) sufficiently small. Then there exist positive real constants \( \rho \) and \( \delta \) such that for all \( t > 0 \)
\[
||y(t) - \pi(z(t))|| = \rho e^{-\delta t} ||y(0) - \pi(z(0))||.
\]

**Lemma 3.** [26] Suppose the system
\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx,
\end{align*}
\]
is stabilizable and detectable, then the system is stabilizable by the dynamic output feedback
\[
\begin{align*}
\dot{u} &= Mx, \\
\dot{x} &= K\dot{x} + Ey,
\end{align*}
\]
where \( M, K \) and \( E \) are appropriate matrices.

**Lemma 4.** [19]. The following are equivalent:

(i). \( (X, f, h) \) is immersed into a finite-dimensional and observable linear system;

(ii). the observation space of \( (X, f, h) \) has finite dimension over \( \mathbb{R}; \)

(iii). there exist an integer \( q \) and a set of real numbers \( a_0, a_1, \ldots, a_{q-1} \) such that \( L^q h(x) = a_0 h(x) + a_1 L_1 h(x) + \cdots + a_{q-1} L_1^{q-1} h(x) \).

**Definition 1.** [17]. In a nonlinear system \( \dot{w} = s(w) \), an initial condition \( w_0 \) is said to be Poisson stable if the flow \( \Phi_t^s(w_0) \) of the vector field \( s(w) \) is defined for all \( t \in \mathbb{R} \) and for each neighborhood \( U \) of \( w_0 \) and for each real number \( T > 0 \), there exists a time \( t_i > T \) such that \( \Phi_t^s(w_0) \in U \), and a time \( t_2 > T \) such that \( \Phi_t^s(w_0) \in U \).

**Definition 2.** [19]. Consider the two following dynamic systems:
\[
\dot{x} = f(x), \quad x \in X,
\]
\[
y = h(x), \quad z \in \mathbb{R}^m,
\]
and
\[
\dot{z} = g(z), \quad z \in \mathbb{Z},
\]
\[
y = l(z), \quad y \in \mathbb{R}^m.
\]
where \( 0 \in X \subset \mathbb{R}^p, \quad 0 \in Z \subset \mathbb{R}^q, \quad p \leq q \). Then system (9) is immersed into system (10) if there exists a smooth mapping \( r : X \rightarrow Z \) satisfying \( r(0) = 0 \) and
\[
\dot{h(x)} = l(z) \Rightarrow h(l(z)) \neq l(h(z)),
\]
such that
\[
\frac{d}{dt} f(x) = g(l(x)),
\]
for all \( x \in X \).

**Definition 3.** The distributed robust formation output regulation of system (7) is solvable, if for all sufficiently small parameter perturbation \( \mu \), the following conditions hold:

a) the closed system (7) is locally asymptotically stable at state equilibrium \( (x, z) = (0, 0) \) when \( w = 0 \);

b) for any initial condition \( (x(0), z(0), w(0)) \) in a sufficient small neighborhood of \( (0, 0, 0) \) such that
\[
\lim_{t \to +\infty} e(t) = 0.
\]

In fact, as pointed out in [24], by treating the exo-system as an active leader, the formulation of distributed robust formation output regulation includes various multi-agent problems as special cases.

1) When without exo-system (i.e., \( w = 0 \)), then the distributed robust formation output regulation problem becomes a leaderless formation problem.

2) If \( r_1 = r_2 = \cdots = r_\ell = 0 \), then the distributed robust formation output regulation problem becomes distributed robust output regulation problem which has been studied by many scholars.

To solve the distributed robust formation output regulation problem, some assumptions are given as follows:

(A1) The leader is globally reachable in \( \mathbb{T} \).

(A2) The point \( w = 0 \) is a stable equilibrium of \( \dot{w} = s(w) \), and each initial condition \( w(0) \) is Poisson stable.

**Remark 2.** In graph \( \mathbb{G} \), some agent may not connect with the other agents and cannot obtain the information too, it is an isolated node. If the leader is not globally reachable in \( \mathbb{T} \), then the agent cannot get the signal from the leader even by the distributed scheme. So (A1) is a basic condition for solving the distributed formation output regulation problem.

3. Main results

Next we will consider the distributed robust formation output regulation problem of the system (1), and give the main results of this paper.

**Theorem 1.** For assumption (A2), assume that the condition (a) in **Definition 3** is fulfilled, then the following statements are equivalent:

(i). The condition (b) in **Definition 3** is fulfilled:
(ii) There exist $C^k(k \geq 2)$ mapping $x = \pi^e(w, \mu)$ with $\pi^e(0, \mu) = 0$, and $z = \sigma(w)$ with $\sigma(0) = 0$, such that

$$
\frac{\partial \pi^e(w, \mu)}{\partial w}(w) = f(\pi^e(w, \mu), \theta(\sigma(w), 0), w, \mu),
$$

$$
\frac{\partial \sigma(w)}{\partial w}(w) = \eta(\sigma(w), 0),
$$

$$
\pi^e(w) - 1 \otimes q(w) = 0.
$$

(11)

**Proof.** Let $w^e = \begin{pmatrix} w \\ \mu \end{pmatrix}$ as the state of the augmented exo-system

$$
w^e = \begin{pmatrix} S(w) \\ 0 \end{pmatrix} = S(w^e).
$$

Then system (6) can be rewritten as follows:

$$
x = f(x, u, w^e),
$$

$$
w^e = S(w^e).
$$

(12)

And the assumption (A2) is satisfied for system (12). Since $f(\cdot, s, w)$ and $q(w)$ are $C^k(k \geq 2)$ functions, we can obtain the linear approximation of the nonlinear system (7) using the Taylor expansion. This approximation system has the following form:

$$
x = (A(\mu) + B(\mu)F_z)x + B(\mu)F_zz - (B(\mu)F_zB_01)\otimes Qw + C(\mu)w + \phi(x, z, w, \mu),
$$

$$
z = G_0x + G_0z - (G_0B_01)\otimes Qw + \phi(x, z, w, \mu),
$$

$$
w = Sw + \phi^e(x, z, w, \mu),
$$

$$
(13)

where

$$
A(\mu) = \frac{\partial f}{\partial x}(0, \mu, 0),
$$

$$
B(\mu) = \frac{\partial f}{\partial u}(0, \mu, 0),
$$

$$
C(\mu) = \frac{\partial f}{\partial w}(0, \mu, 0),
$$

$$
F_z = \frac{\partial \phi}{\partial z}(0, 0),
$$

and

$$
G_T = \frac{\partial \phi}{\partial e_{\mu}}(0, 0),
$$

$$
G_z = \frac{\partial \phi}{\partial e_{z}}(0, 0),
$$

$$
S = \frac{\partial S}{\partial w}(0).
$$

Therefore, for sufficiently small parameter perturbation $\mu$, as it is assumed that the condition (a) in Definition 3 is fulfilled, then the eigenvalues of the following matrix

$$
\begin{pmatrix}
A(\mu) + B(\mu)F_zH & B(\mu)F_z \\
G_TH & 0
\end{pmatrix}
$$

have negative real part. Moreover, from assumption (A2), we can know the eigenvalues of $S$ are on an imaginary axis. Therefore, from Lemma 1, there exist continuously differentiable functions $\pi^e(w, \mu)$ and $\sigma(w)$ with $\pi^e(0, \mu) = 0, \sigma(0, \mu) = 0$

$$
\begin{pmatrix}
\pi^e(w, \mu) \\
\sigma(w)
\end{pmatrix} = \begin{pmatrix}
\pi^e(0, \mu) \\
\sigma(0, \mu)
\end{pmatrix},
$$

is a center manifold for system (7). Firstly, we prove $\|i\| = ii$.

Substituting $x = \pi^e(w, \mu)$, $z = \sigma(w)$ into the first two equations of the system (7), we have

$$
\frac{\partial \pi^e(w, \mu)}{\partial w}(w) = f(\pi^e(w, \mu), \theta(\sigma(w), e_\mu(w)), w, \mu),
$$

$$
\frac{\partial \sigma(w)}{\partial w}(w) = \eta(\sigma(w), e_\mu(w)),
$$

(14)

with $e_\mu(w) = H\pi^e(w, \mu) - (B_01\otimes q(w)).$

As

$$
\frac{\partial \pi^e(w, \mu)}{\partial w}(w) = \begin{pmatrix}
\frac{\partial \pi^e(w, \mu)}{\partial w}(w) \\
0
\end{pmatrix},
$$

$$
\frac{\partial \sigma(w)}{\partial w}(w) = \begin{pmatrix}
0 \\
\frac{\partial \sigma(w)}{\partial w}(w)
\end{pmatrix},
$$

thus we have

$$
\frac{\partial \pi^e(w, \mu)}{\partial w}(w)S(w) = f(\pi^e(w, \mu), \theta(\sigma(w), e_\mu(w)), w, \mu),
$$

$$
\frac{\partial \sigma(w)}{\partial w}(w)S(w) = \eta(\sigma(w), e_\mu(w)).
$$

Suppose $w_0$ is any initial condition of $w^e = \pi^e(w, \mu)$, then for sufficiently small $\mu$, the trajectory $(x(t), z(t), w^e(t))$ of system (7) satisfies $(x(0), z(0), w^e(0)) = (\pi^e(0, \mu), \sigma(0, \mu), w^e(0))$. Thus, by assumption (A2), for any $\epsilon > 0$ and $T > 0$, there exists some $t > T$ such that $|(x(T), z(T), w^e(T)) - (x(0, \mu), \sigma(0, \mu), w^e(0))| < \epsilon$.

So every trajectory on the center manifold cannot converge to zero. Thus, condition (b) in Definition 3 is fulfilled only if $e = x - 1 \otimes q(w) = 0$ i.e. $x(0) - 1 \otimes q(w) = 0$. We also have $\pi(0) = x = q(0)$, then from the virtual regulated error (2), we can have $e_\mu(0) = 0$ and obviously, so Eq. (11) holds.

(12)

From the third equation of (11), we have

$$
e(t) = \frac{\partial \pi^e}{\partial w}(0) - \frac{\partial q(0)}{\partial w} = [\pi^e(w, \mu) - \sigma(w, \mu)] = 0.
$$

On the other hand, from the first two equations of (11), the mapping

$$
\begin{pmatrix}
\pi^e(w) \\
\sigma(w)
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

is a center manifold for system (7). Then from Lemma 2, for any $\rho > 0, \delta > 0$ and all $(\rho(0), 0)$ which is sufficiently close to the equilibrium state $(x, w) = (0, 0)$, we have $|\rho(T) - \pi^e(w(T))| = \rho(T) - \pi^e(w(T))| < \rho\delta$. Thus, by assumption (A2), we can know the eigenvalues of $S$ are on an imaginary axis. Therefore, from Lemma 1, there exist continuously differentiable functions $\pi^e(w, \mu)$ and $\sigma(w)$ with $\pi^e(0, \mu) = 0, \sigma(0, \mu) = 0$

$$
\pi^e(w, \mu) - 1 \otimes q(w) = 0,
$$

(15)

and for $(w, \mu)$ the autonomous system with output

$$
w^e = \pi^e(w^e),
$$

$$
u = c(\mu, w^e),
$$

(16)

is immersed into

$$
z = \theta(z),
$$

$$
u = \gamma(z),
$$

(17)

with $\theta(0) = 0, \gamma(0) = 0$. For some choice of the matrix $N$ and the matrices

$$
G_T = \begin{pmatrix}
\partial \theta \\
\partial z
\end{pmatrix},
$$

$$
F_z = \begin{pmatrix}
\partial \gamma \\
\partial z
\end{pmatrix},
$$

such that the pair

$$
\begin{pmatrix}
A(\mu) \\
0
\end{pmatrix},
$$

$$
\begin{pmatrix}
B(\mu)F_z \\
0
\end{pmatrix}
$$

is stabilizable, and the pair

$$
\begin{pmatrix}
H \\
0
\end{pmatrix},
$$

$$
\begin{pmatrix}
A(\mu) \\
B(\mu)F_z
\end{pmatrix}
$$

is detectable.

**Proof.** Necessity. In graph $G$, if the leader is not globally reachable, some agent may not connect with the other agents and cannot obtain the information from the leader simultaneously, it is an
isolated node, for this agent, we have $a_j = 0 (j = 1, 2, \ldots, N)$ and $b_j = 0$, then the virtual regulated error $e_n = \sum_{j \neq k_i} a_j (e_j - e_i) + b_i e_i = 0$.

Then the distributed controller based on the virtual regulated error becomes meaningless. So from assumption 1, the controller based on the virtual regulated error of multi-systems can be rewritten in the form of (5). Suppose that a controller of the form (5) solves the problem of output regulation. Then, by Theorem 1, for sufficiently small $\mu$, there exist mappings $\mathbf{x} = \pi(\mathbf{w}, \mu), Z = \sigma(\mathbf{w})$ with $\pi(0, \mu) = 0, \sigma(0) = 0$, such that (11) is satisfied. Setting $c^*(\mathbf{w}, \mu) = \sigma(\pi(\mathbf{w}, 0)), \theta$ substituting it into (11) yields (15). So condition (15) is satisfied for $\pi^*(\mathbf{w}, \mu)$ and $c^*(\mathbf{w}, \mu)$.

Let $\gamma(z) = \theta(z, 0), \delta(z) = \eta(z, 0)$, then $\gamma(z)$ and $\delta(z)$ satisfy

$$
\frac{\partial \sigma(w)}{\partial w} s(w) = \theta(\sigma(w)),
$$

$$
c^*(w, \mu) = \gamma(\sigma(w)).
$$

Thus from Definition 2, it can be seen that the autonomous system (16) is immersed into system (17). And we have

$$
\frac{\partial \mu}{\partial z} = 0, \mu = 0
$$

$$
\frac{\partial \mu}{\partial z} = 0, \mu = 0
$$

According to the hypothesis of the theorem, i.e., the distributed robust formation output regulation problem is solvable; therefore, the eigenvalues of the matrix

$$
\begin{pmatrix}
A(\mu) & B(\mu)F_{2H} \\
G_{2H} & G_z
\end{pmatrix}
$$

all have negative real parts. Observe that

$$
\begin{pmatrix}
A(\mu) & B(\mu)F_{2H} \\
G_{2H} & G_z
\end{pmatrix}
= \begin{pmatrix}
A(\mu) & 0 \\
G_{2H} & G_z
\end{pmatrix} + \begin{pmatrix}
B(\mu) & \mu F_{2H} \\
0 & F_{2H} & F_{2z}
\end{pmatrix}.
$$

Let $N = G_z$, then the pair

$$
\begin{pmatrix}
A(\mu) & 0 \\
N & G_z
\end{pmatrix}
$$

is stabilizable.

Similarly, from

$$
\begin{pmatrix}
A(\mu) & B(\mu)F_{2H} \\
G_{2H} & G_z
\end{pmatrix}
= \begin{pmatrix}
A(\mu) & B(\mu)F_{2H} \\
0 & G_z
\end{pmatrix} + \begin{pmatrix}
B(\mu) & \mu F_{2H} \\
0 & F_{2H} & F_{2z}
\end{pmatrix}.
$$

we can know that the pair $(H, 0)$, $\begin{pmatrix}
A(\mu) & B(\mu)F_{2z} \\
0 & G_z
\end{pmatrix}$ is detectable.

Sufficiency. From the conditions, the pair $\begin{pmatrix}
A(\mu) & 0 \\
N & G_z
\end{pmatrix}$ is stabilizable when choosing the proper matrix $N$, and the pair $(H, 0), \begin{pmatrix}
A(\mu) & B(\mu)F_{2z} \\
0 & G_z
\end{pmatrix}$ is detectable, so that the pair

$$
\begin{pmatrix}
A(\mu) & B(\mu)F_{2H} \\
N & G_z
\end{pmatrix}
$$

is also stablizable and the pair

$$
(H, 0), \begin{pmatrix}
A(\mu) + B(\mu)F_{2H} & B(\mu)F_{2z} \\
NH & G_z
\end{pmatrix}
$$

is also detectable. Then, by Lemma 3, there exist matrices $M, K, E$ such that the matrix

$$
\Psi = \begin{pmatrix}
A(\mu) + B(\mu)F_{2H} & B(\mu)F_{2z} & B(\mu)M \\
NH & G_z & 0 \\
E(H) & 0 & K
\end{pmatrix}
$$

has all eigenvalues with negative real part.

Now consider the controller as follows:

$$
u = Mz_0 + \gamma(z_1) + F_e e_v,
$$

$$z_0 = Kz_0 + Ne_v,
$$

$$z_1 = \theta(z_0) + Ne_v.
$$

When $w = 0$, for sufficiently small $\mu$, at $(x, z_0, z_1) = (0, 0, 0)$, the Jacobian matrix of the closed system has the form

$$
\Psi^* = \begin{pmatrix}
A(\mu) + B(\mu)F_{2H} & B(\mu)M & B(\mu)F_{2z} \\
EH & K & 0 \\
NH & 0 & G_z
\end{pmatrix}
$$

Matrix $\Psi$ in (21) has all eigenvalues with negative real part, the eigenvalues of matrix (23) also have negative real part. Therefore, the requirement (a) in Definition 3 is satisfied for the designed controller (22).

Moreover, by conditions of the theorem and Lemma 4, there exist mappings $\mathbf{x} = \pi^*(\mathbf{w}, \mu), u = c^*(\mathbf{w}, \mu)$ and $z_1 = \pi(\mathbf{w})$ such that (15) holds and

$$
\frac{\partial \tau}{\partial w}(w) = \theta(\pi(w)),
$$

$$c^*(w, \mu) = \gamma(\pi(w)).
$$

Let $\begin{pmatrix}
z_0 \\
z_1
\end{pmatrix} = \pi(\mathbf{w}) = \begin{pmatrix}
0 \\
\pi(w)
\end{pmatrix}$. This shows that the sufficient conditions of Theorem 1 are satisfied. Then, by Theorem 1, the requirement (b) in Definition 3 is also fulfilled. So it can be concluded that the distributed robust output regulation problem of system (1) is solvable. This completes the proof.

Remark 3. In fact, the dynamic controller (22) consists of two parts in parallel:

$$u = \gamma(z_1),
$$

$$z_1 = \theta(z_0) + Ne_v,
$$

and

$$u = Mz_0 + F_e e_v,
$$

$$z_0 = Kz_0 + Ne_v.
$$

The first part, due to the immersion, is called the internal model or servo-compensator and can asymptotically recover $c^*(w, \mu)$, whereas the second part is called the stabilizing compensator and renders the invariant zero-error manifold locally attractive.

4. Example

In this section, we will provide a simple example to illustrate the effectiveness of the results. Consider a multi-agent system consisting of four follower agents with graph described by the Laplacian and the diagonal matrix for the interconnection between
the leader and the follower agents

\[ L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

The dynamics of the followers are described as

\[ x_i = \sin x_1 + u_1, \]
\[ x_2 = \mu x_2 - x_2^3 + u_2, \]
\[ x_3 = \mu x_3^2 + u_3, \]
\[ x_4 = u_4 \tag{24} \]

and the exo-system is modeled as follows:

\[ w_1(t) = w_2(t), \]
\[ w_2(t) = -w_1(t), \]
\[ y_0 = w_1(t). \tag{25} \]

Assume that the formation vector is

\[ r_1 = (4 \ 0)^T, \ r_2 = (0 \ 8)^T, \ r_3 = (-4 \ 0)^T, \ r_4 = (0 \ -8)^T. \]

Let \( e_i(t) = x_i(t) - r_i - y_{0i}, \ i = 1, 2, 3, 4. \)

In fact, it can be directly verified that the solution of the regulator Eq. (15) for this system is

\[ x^w(w, \mu) = (w_1 \ w_1 \ w_1 \ w_1)^T, \]
\[ c_i^w(w, \mu) = (w_2 - \sin w_1 \ w_2 - \mu w_1 + w_1^3 \ w_2 - \mu w_1^2 \ w_2)^T \]

Furthermore, the results mentioned in reference [19] show that the autonomous system with outputs (16) can be immersed into the following system:

\[ \dot{z} = g(z) = \begin{bmatrix} G_{z1} & 0 & 0 & 0 \\ 0 & G_{z2} & 0 & 0 \\ 0 & 0 & G_{z3} & 0 \\ 0 & 0 & 0 & G_{z4} \end{bmatrix} z = G_z z. \]

\[
F_e = \begin{bmatrix}
0.1796 & -1.5711 & 1.3002 & -0.8742 & 1.0023 & 1.2213 & -1.5562 & -0.0912 \\
-1.4746 & -1.0213 & 1.7132 & -0.1943 & 0.5536 & 0.3967 & -0.4291 & -0.0166 \\
0.2410 & -0.4923 & 0.9820 & 0.4916 & 0.7162 & -2.0325 & -1.2448 & -1.3039 \\
-0.5236 & 1.0486 & 1.2569 & -1.8233 & -2.3042 & -0.4535 & -0.5022 & 1.4437 \\
-1.4203 & 0.1914 & 0.6572 & 0.2653 & -0.6165 & -0.5198 & -0.1295 & 1.0930 \\
-0.4236 & -1.8233 & -0.9564 & 0.1233 & 0.2471 & 0.8930 & 1.1432 & -1.5293 \\
1.6925 & 0.3964 & 0.0381 & 0.4591 & 0.1726 & -0.1150 & 0.0344 & 1.0032 \\
-0.6416 & 0.0937 & -0.9218 & -1.8233 & -0.6920 & 0.3049 & -0.2538 & 0.6810 \\
\end{bmatrix}
\]

Such that matrix (23) has eigenvalues with all negative real parts. Therefore, using Theorem 2, the output regulation problem of the system in this example is solved.

Consider some initial values of the system randomly. Then the trajectories of regulated outputs for the four follower agents are shown in Fig. 1. Through the simulation results in Fig. 1, we can see that the regulation error \( e_i \) of the four follower agents asymptotically converges to zero. And from Fig. 2, we know the followers acquire a pre-defined geometric shape. Therefore, this simulation example demonstrates the effectiveness of the main results in this paper.
5. Conclusions

In this paper, we studied the robust output regulation problem to the formation control of multi-agent systems with general nonlinear systems. To solve this problem, we designed a nonlinear distributed dynamic feedback controller, which can make the agents of the multi-agent system considered asymptotically track the reference or reject disturbance. By solving the regulator equations, the distributed robust output regulation formation control problem can be solved. Finally, a numerical example is provided to illustrate the effectiveness of the proposed results.

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Reference


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