Synchronization between two general complex networks with time-delay by adaptive periodically intermittent pinning control

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1. Introduction

A complex network is a large set of interconnected nodes and has many applications in almost all the fields of the real world, such as the neural networks, the cellular and metabolic networks, the Word Wide Web, the electrical power grids and the social networks [1–4]. In the former literature, several kinds of network models have been proposed for the purpose of describing the real world more realistic [5,6], such as regular networks, random networks [6], small world networks [1] and scale free networks [3]. Besides these studies of topology structures, synchronization is a significant dynamical behavior of the dynamical elements in the complex networks and has been widely investigated [7–9]. In the coupling networks, there are several kinds of synchronization, for example, complete synchronization [10], phase synchronization [11], cluster synchronization [12,13], partial synchronization [14] and so on. Different from these categories, the phenomenon of synchronization can also be classified into “inner synchronization” [7,8] and “outer synchronization” [15–18]. “Inner synchronization” means a collective behavior of all the nodes within a network, while “outer synchronization” refers to the synchronization occurring between two or more coupled networks regardless of happening of the inner synchronization. In Nature, there are so many examples that can be taken to illustrate the phenomenon of outer synchronization, for instance, the infectious disease spreads between different communities, the avian influenza spreads among domestic and wild birds, and the different species development in balance. All these challenging topics show the great importance of researching outer synchronization between coupled networks. Thus, the outer synchronization has attracted more and more attention.

In some cases, different complex networks can achieve outer synchronization by themselves. But, there also exists the situation that the networks with identical system parameters cannot synchronize with each other by themselves for different initial values. Thus, many kinds of control techniques have been adopted to make the networks achieve synchronization [19–25]. Among these methods, a pinning control is a special control method of adding controllers to partial of the nodes rather than all of the nodes [26]. The pinning control not only simplifies the coupling topological structure, but also saves the cost [16] through reducing the number of controllers. Thus, its status in engineering application has become more and more important.

In the process of controlling, the signal will become weak due to diffusion, so it needs some external control until the strength of the signal reaches an upper level. After that, the external control can be removed in order to reduce the cost. This kind of discontinuous control method is different from the continuous control and is named as an intermittent control. Usually, the control time of intermittent is periodic, and in every period, the time with a controller is denoted as control time and the rest is denoted as rest time (see Fig. 1) [27]. From the perspective of economic costs, the method of pinning control will reduce the number of controllers in the complex networks and the periodically intermittent control will decrease the working time. Therefore, if we combine two kinds of control methods together, the control cost will greatly be saved [28–33].
In this paper, we focus on the problem of outer synchronization between two generalized complex dynamical networks with time delay. By using the adaptive pinning periodically intermittent control, some novel synchronization criteria will be obtained by the Lyapunov stability theory. Numerical simulations are also presented to show the effectiveness of proposed theoretical results.

The rest of this paper is organized as follows. In Section 2, some general driver and response complex dynamical network models are introduced, and some necessary preliminaries are given. In Section 3, based on the Lyapunov stability theorem, some pinning controllers are designed to ensure that the driver and response systems with time delay achieve outer synchronization under different situations. In Section 4, some numerical simulations are given to verify the effectiveness of proposed theoretical results and Section 5 gives the conclusion of the paper.

2. Network model and preliminaries

Here we consider a complex network consisting of N identical linearly and diffusively coupled nodes, and every node in the network is an n dimensional dynamical unit. Then the network model of the drive system is denoted as

\[ x_i(t) = F(t, x_i(t), x_i(t - \tau)) + c \sum_{j=1}^{N} g_{ij} A x_j(t) + u_i(t), \quad i = 1, 2, \ldots, N, \]  

where \( x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n \) is the state variable of the \( i \)-th node, \( F(t, x_i(t), x_i(t - \tau)) = f_1(t, x_i(t), x_i(t - \tau)), f_2(t, x_i(t), x_i(t - \tau)), \ldots, f_n(t, x_i(t), x_i(t - \tau)) \) \( \in \mathbb{R}^n \). \( f_i(t, x, y) : \mathbb{R}^{2n} \to \mathbb{R}^n \) is a nonlinear vector valued function describing the dynamics of nodes and \( c > 0 \) is the coupling strength of the whole network. The matrix \( G = (g_{ij}) \in \mathbb{R}^{n\times n} \) is the outer coupling configuration matrix, in which \( g_{ij} \in \mathbb{R} \) is defined as follows: if there is a coupling from node \( i \) to node \( j (i \neq j) \), \( g_{ij} > 0 \); otherwise, \( g_{ij} = 0 \). At the same time, the diagonal elements of \( G \) are defined as \( g_{ii} = -\sum_{j=1, j \neq i}^{N} g_{ij} \). The inner coupling matrix \( A \) denotes the inner coupling relationship between every two nodes.

Compared with the drive system mentioned above, the response complex network is denoted as

\[ y_i(t) = f(t, y_i(t), y_i(t - \tau)) + c \sum_{j=1}^{N} g_{ij} A y_j(t), \quad i = 1, 2, \ldots, N, \]  

where \( y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T \in \mathbb{R}^n \) denotes the state variables of the response system, and other parameters involved in the system (2) all have the same meanings with the corresponding parameters in system (1).

Remark 1. In the drive and response systems above, the outer coupling configuration matrix \( G \) need not to be symmetric or irreducible.

In this paper, we assume that \( \|A\|_2 = \alpha > 0 \) and \( \rho_{\min} \) denotes the minimum eigenvalue of matrix \( (A + A^T)/2 \). \( G \) is a modifying matrix of \( G \) via replacing the diagonal elements \( g_{ii} \) by \((\rho_{\min}/\alpha)g_{ii} \). Choose the matrix \( \tilde{G} = (\tilde{G} + \tilde{G}^T)/2 \) and its eigenvalues are expressed as \( \lambda_1 \geq \lambda_2 \geq \ldots \lambda_N \).

In the following, some basic definition, lemmas, and assumption will be given.

Definition 1. The drive system (1) and the response system (2) are defined to achieve synchronization, if

\[ x_i(t) - y_i(t) \to 0, \quad t \to \infty, \quad i = 1, \ldots, N. \]  

Assumption 1. If there exists constants \( k_d > 0 \), the nonlinear function \( f_i(t, x(t), x(t - \tau)) \) satisfies the uniform Lipschitz condition with respect to time \( t \),

\[
|f_i(t, x_i(t), x_i(t - \tau)) - f_i(t, y_i(t), y_i(t - \tau))| 
\leq \sum_{i=1}^{N} k_0(|x_i(t) - y_i(t)| + |x_i(t - \tau) - y_i(t - \tau)|) 
\]  

where \( 1 \leq r \leq n \).

Lemma 1 (Schur complement, Boyed et al. [34]). The following linear matrix inequality (LMI)

\[
\begin{pmatrix}
A(x) & B(x) \\
(B(x))^T & C(x)
\end{pmatrix} > 0,
\]

where \( A(x) = (A(x))^T \), \( C(x) = (C(x))^T \) is equivalent to one of the following conditions:

(a) \( A(x) > 0 \) and \( C(x) - B(x)A(x)^{-1}B(x) > 0 \); 
(b) \( C(x) > 0 \) and \( A(x) - B(x)C(x)^{-1}B(x) > 0 \).

Lemma 2 (Halany [35]). Let \( w : [\mu - \tau, \infty) \to [0, \infty) \) be a continuous function such that

\[ w(t) \leq -aw(t) + b \max w_i \]

holds for \( t \geq \mu \). If \( a > b > 0 \), then

\[ w(t) \leq \max w_i \] \[ \exp(-\gamma(t-\tau)), \quad t \geq \mu \]

where \( \max w_i = \sup_{t-\tau \leq \theta \leq t} w_i(\theta) \), and \( \gamma > 0 \) is the smallest real root of the equation

\[ a - \gamma - b \exp(\gamma \tau) = 0. \]

Lemma 3 (Xia and Cao [31]). Let \( w : [\mu - \tau, \infty) \to [0, \infty) \) be a continuous function such that

\[ w(t) \leq aw(t) + b \max w_i \]

holds for \( t \geq \mu \). If \( a > b > 0 \), then

\[ w(t) \leq \max w_i \] \[ \] \[ \exp(a + b(t - \tau)), \quad t \geq \mu \]

where \( \max w_i = \sup_{t-\tau \leq \theta \leq t} w_i(\theta) \).

3. Main results

In this part, in order to realize outer synchronization between the drive and response systems by adaptive pinning periodically intermittent control, some controllers are needed to add on partial nodes of the network. Denote the error expression as

\[ e_i(t) = x_i(t) - y_i(t), \quad i = 1, 2, \ldots, N. \]

Here, select the first \( l \) nodes to be pinned and the adaptive pinning periodically intermittent control \( u_i(t) \) is designed as follows:

\[
u_i(t) = \begin{cases} 
-k_i \exp(t), & 1 \leq i \leq l, \quad t \in [mT, mT + h), \\
0, & l + 1 \leq i \leq N, \quad t \in [mT, mT + h), \\
0, & 1 \leq i \leq N, \quad t \in [mT + h, (m+1)T). 
\end{cases}
\]
the updating laws
\[
\begin{align*}
k_i(t) &= \left\{ \begin{array}{ll}
\alpha_i \exp(a_1 t) \| e_i(t) \|^2, & t \in [mT, mT + h), \\
0, & t \in [mT + h, (m+1)T)
\end{array} \right.
\end{align*}
\]
where \( \alpha_i(i = 1, 2, ..., N) \) and \( a_i \) are positive constants, \( k_i(0) > 0 \) \((i = 1, 2, ..., l)\) are initial value and \( k_i(mT + 1) = k_i(mT + h), m = 0, 1, 2, ... \). \( T > 0 \) denotes the control period and the work time \( h \) satisfies \( 0 < h < T \).

Then, based on the adaptive pinning periodically intermittent controllers (7) and the error expression (6), the error systems can be written as
\[
\dot{e}_i(t) = \left\{ \begin{array}{ll}
F_i(t, (t - \tau)) + c \sum_{j = 1}^{N} g_{ij} \dot{e}_j(t) - k_i(t)e_i(t), & 1 \leq i \leq l, \\
\overline{F}_i(t, (t - \tau)) + c \sum_{j = 1}^{N} g_{ij} \overline{e}_j(t), & l + 1 \leq i \leq N.
\end{array} \right.
\]
where \( \overline{F}(t, (t - \tau)) = F(t, x_i(t), x_i(t - \tau)) - F(t, y_i(t), y_i(t - \tau)). \)

When \( t \in [mT, mT + h), \)
\[
\dot{e}_i(t) = \left\{ \begin{array}{ll}
F_i(t, (t - \tau)) + c \sum_{j = 1}^{N} g_{ij} \dot{e}_j(t) - k_i(t)e_i(t), & 1 \leq i \leq l, \\
\overline{F}_i(t, (t - \tau)) + c \sum_{j = 1}^{N} g_{ij} \overline{e}_j(t), & l + 1 \leq i \leq N.
\end{array} \right.
\]

**Theorem 1.** Suppose that \( \tau \leq h \) and \( \tau \leq T - h \), and choose \( h = R_1 T, r = R_2 T \). Then, if there exist the positive constants \( a_1, a_2, \) and \( k \) such that
\[
\begin{align*}
Q &= \left( p + \frac{1}{2} a_1 \right) h_n + ca_2 g^2 - D \leq 0, \\
p &- \frac{1}{2}(a_2 - a_1) + ca_2 \lambda_1 \leq 0, \\
\gamma(R_1 - R_2) - (a_2 - a_1) + q(1 - R_1) > 0,
\end{align*}
\]
where
\[
D = \text{diag}(k_1, ..., k_l, 0, 0, ..., 0),
\]
p = \max_{1 \leq i \leq n} \lambda_i = \max_{1 \leq i \leq n} \| (1/2) \sum_{j = i}^{n} (2k_{ij} + k^{1, (i-j)}) \|, \quad q = \max_{1 \leq i \leq n} \lambda_1 = \max_{1 \leq i \leq n} \sum_{j = i}^{n} k_{ij}^{1, (i-j)}, \quad a_2 > a_1 > q, \quad \gamma > 0 \text{ is the smallest real root of the equation } a_1 - \gamma - q \exp(\gamma t) = 0. \text{ Then, the drive system (1) and the response system (2) achieve outer synchronization under the adaptive pinning periodically intermittent controllers (7).}

**Proof.** In order to verify the conclusion of **Theorem 1**, it is necessary to construct a Lyapunov function as
\[
V(t) = \frac{1}{2} \sum_{i = 1}^{N} e_i^2(t)e_i(t) + \frac{1}{2} \sum_{i = 1}^{l} \exp(-a_1 t) \frac{k_i(t) - k_i}{a_1}. \tag{12}
\]

When \( mT \leq t < mT + h \), using the Cauchy inequality and the conditions in **Theorem 1**, the derivative of \( V(t) \) with respect to time \( t \) along the trajectory of the error system (9) can be calculated as follows:
\[
\dot{V}(t) = \sum_{i = 1}^{N} e_i^2(t)\dot{e}_i(t) - \frac{a_1}{2} \sum_{i = 1}^{N} \exp(-a_1 t) \frac{k_i(t) - k_i}{a_1}
+ \frac{1}{2} \sum_{i = 1}^{l} \exp(-a_1 t) \frac{k_i(t) - k_i}{a_1}
= \sum_{i = 1}^{N} e_i^2(t)(F_i(t, x_i(t), x_i(t - \tau)) - F_i(t, y_i(t), y_i(t - \tau)))
+ c \sum_{i = 1}^{N} \sum_{j = 1}^{N} g_{ij} e_j^2(t)A_i^2(t) - \frac{1}{2} \sum_{i = 1}^{l} k_i(t)e_i(t)
\]

Where \( e(t) = (\| e_1(t) \|_2, \| e_2(t) \|_2, ..., \| e_N(t) \|_2)^T, p = \max_{1 \leq r \leq n} (1/2) \sum_{i = r}^{n} (2k_{nr} + k^{1, (i-r)}), \) \( q = \max_{1 \leq i \leq n} \lambda_1 = \max_{1 \leq i \leq n} \sum_{j = i}^{n} k_{ij}^{1, (i-j)} \), \( a_1 > q, \) \( V(t) \) is satisfied of (12) and thus, according to **Lemma 2**, we have
\[
V(t) \leq \max_{mT - \tau \leq \theta \leq mT} V(\theta) \exp(-\gamma(t - mT)), \tag{14}
\]
where \( \gamma \) is the smallest real root of the equation \( a_1 - \gamma - q \exp(\gamma t) = 0. \)
When \( r_{\text{max}} \) is the smallest real root of the equation \( \alpha_{\gamma} \gamma T - (a_{1} + a_{2} + \eta) \gamma (h - \tau) = 0 \), the drive system (1) is synchronous with the response system (2) by the adaptive pinning periodically intermittent controllers (7).

Letting \( \Gamma = p + \alpha a_{1} \) and selecting a positive constant \( a_{2} \) which satisfy the condition of \( a_{3} = 2G + a_{1} \), then the second inequality of (21) in Corollary 1 holds. The fourth inequality of (21) can be reduced \( \gamma (R_{1} - R_{2}) > 0 \) if \( \max_{\gamma_{ij}} \gamma_{ij} > 0 \) is the smallest real root of the equation \( a_{1} - \gamma - q \exp (\gamma T) = 0 \). Then the drive system (1) is synchronous with the response system (2) by the adaptive pinning periodically intermittent controllers (7) and the updating laws (8).

4. Numerical simulation

In this section, the Chua oscillator with time delay is used as an uncoupled node in the drive and response systems to show the
systems for 0

Fig. 4. Synchronous errors $e_i(t), i=1, 2, \ldots, 10$ between the drive and response systems for $0 \leq t \leq 1.5, T=0.2$, and $T=2$.

Fig. 5. Synchronous errors $e_i(t), i=1, 2, \ldots, 10$ between the drive and response systems for $0 \leq t \leq 1.5, T=0.2$, and $T=2$.

Fig. 6. Adaptive control gain of controllers.
and
\[ f_3(t, x_i(t), x_j(t) - r) - f_3(t, y_i(t), y_j(t) - r)) \]
\[ \leq \beta \langle x_i(t) - y_i(t) + \beta x_i(t) - y_i(t) + \omega(x_i(t) - y_i(t)). \]

Calculate the value of the parameters, then we have \( f_1 = \alpha (1 + m_2) + (1/2)\alpha (m_1 - m_1) = 5.416, f_2 = 10, f_3 = 0, f_{21} = f_{22} = f_3 = 1, f_3 = \beta = 19.53, f_3 = w = 0.1636, \) and \( q = 1.2. \) Thus, the value of \( p \) and \( q \) is \( p = 2.22284 \) and \( q = 30.53. \)

The coupling configuration matrices of the drive and response systems are chosen as a BA scale model. The parameters of BA model are given by \( m_0 = m = 3, N = 10. \) In the simulations, we add the adaptive periodic intermittent controllers to the first six nodes. According to the coupling matrix, we known that \( \lambda = 0.6790, \alpha = 12. \) and \( \lambda_{\text{max}} = -1.4999, \) then we choose the parameters \( T = 0.2, a_1 = 60, a_2 = 162, \) the value of work time \( h \) will be obtained as \( 0.18 \) and the conditions in Corollary 1 are satisfied. Use the same method, we could obtain the result that the value of work time \( h \) will become larger when control period \( T = 2. \) Therefore, there would be some effect on control cost when the control period becomes larger.

Select the initial values of the nodes in the complex networks randomly, then the errors of the corresponding nodes in the drive and response systems are shown in Figs. 3, 4 and 5. From the simulations, we could see that there is nearly no change for the time of synchronization errors that tend to zero between drive and response systems when \( T = 2. \)

**Remark 2.** In the former literatures [31, 32], Xia and Cai studied the inner synchronization of delayed dynamical nodes via linear periodically pinning intermittent control. In that case, the control gain \( k \) obtained in [31, 32] needed to satisfy \( k > \lambda_{\text{max}} (E-BQ^{-1}) \) which will be larger than the needed values for practical problems. If we adopt the control method in [31, 32], the control gain should satisfy \( k > 459.4634. \) While, in this paper, the numerical simulation of adaptive control gain in Fig. 6 illustrates that our results are less conservative and more practically applicable than it in [31, 32].

5. Conclusions

In this paper, we have investigated the problem of outer synchronization between the drive and response systems with time delay dynamical nodes by means of adaptive pinning periodically intermittent controllers. Based on the Lyapunov stability theory, the adaptive control technique and the differential inequality method, some synchronous criteria have been derived analytically. At last, both the theoretical and numerical analysis illustrate the effectiveness of the proposed control methodology.

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References

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